1 Check of completeness of the Bethe Ansatz

The functions \texttt{NSolve}, \texttt{Union}, \texttt{Sort}, \texttt{Select}, \texttt{Length}, \texttt{Apply@@}, \texttt{Map(/@)}, \texttt{ReplacePart}, \texttt{N}, \texttt{Chop}, \texttt{Round}, \texttt{Expand}, \texttt{FreeQ}, \texttt{Mod} may be useful for doing this exercise.

In this exercise on the example of two magnon solutions we will check that Bethe Ansatz gives all the eigenstates of the hamiltonian

1. How many two-particle states are there in the chain of the length \(L\)?

2. Since Hamiltonian is \(sl(2)\) invariant, each eigenstate of the hamiltonian belongs to some representation of \(sl(2)\). How many 2-particle eigenstates are highest weight vectors of the representation?

Any regular solution of the Bethe Ansatz equations (BAE) (a solution with all rapidities being finite) gives us the eigenstate which is a highest weigh vector. All the other eigenstates can be obtained by application of raising operator.

3. Use the function \texttt{NSolve} to solve the system of BAE

\[
\left( \frac{u + \frac{i}{2}}{u - \frac{i}{2}} \right)^L = \prod_{j=1}^{M} \frac{u_k - u_j + i}{u_k - u_j - i}
\]

for \(M = 2\) and \(L = 6, 7\).

4. Find physically meaningful solutions (with not coinciding rapidities).

\textit{Hint.} The combination \texttt{Union@@(Sort/@ ...)} will remove all identical solutions.

\textit{Hint.} You can use the command \texttt{Select} to leave only the physically meaningful solution.

\textit{Hint} To perform the check that two numbers coincide you have to \texttt{Round} them, otherwise \texttt{Mathematica} will treat them as different. You will need also to control the \texttt{WorkingPrecision} (Option for \texttt{NSolve}) to get a solution with sufficient precision.

5. As a guideline that you are on the right way we give the solution for \(M = 2\) and \(L = 5\):

\[
\text{solu} = \{\{-0.5,0.5\},\{-0.436885,0.0450286\},\{-0.317019-0.502498 \sqrt{\text{ImaginaryI}}, -0.317019+0.502498 \sqrt{\text{ImaginaryI}}\},\{-0.0450286,0.436885\},
,\{0.317019-0.502498 \sqrt{\text{ImaginaryI}},0.317019+0.502498 \sqrt{\text{ImaginaryI}}\}\}
\]

You can use the expression

\[
(\text{Plus@@(1/(#^2+1/4)))}&/@\text{solu}
\]

to find the values of energy for each eigenstate. This command is a useful example of usage of \texttt{Apply (@@), Map (/@)} and pure function definition \((f[#]&\).

6. If you do everything correctly then the obtained number of solutions for \(L = 6\) will be smaller by one then the predicted in 2) number. What is the missing solution? To answer this question you have to apply \texttt{NSolve} to the Bethe equations written in the form which does not contain denominators.
7. Construct the wave function using the obtained solution. Check that $H - E$ is annihilated on the constructed states. Use the lecture notes for the definition of hamiltonian and construction of the states.

Since we solve the equations numerically, $H - E$ will not give zero but some small number. You can use for example

$$\ldots/.x_?NumberQ\to\text{If}[\text{Abs}[x]<10^{-5},0,x]$$

to check that the result is zero with a five digits accuracy.

8. To increase the accuracy of the obtained zero you should significantly increase the precision of the equations' solution. What solutions of BAE require the high precision in calculations? Why?

9. Check with 4-digit accuracy that the solutions of BAE for $L = 9$ are annihilated by $H - E$.

**10 We come back to an exceptional solution (see 6)). For even values of $L$ it has a physical meaning while for odd values of $L$ it has not. To treat it correctly you have to introduce some regularization. Find the energy of the exceptional solution and construct the associated eigenstate.

*Hint.* To get the physical intuition about how the eigenstate looks like, it is good to answer the question: what are the momenta of the particles? What is the total momenta (again, the regularization is needed for this calculation)? Using this intuition you can even guess the answer.