1 The rank one sectors

Particularly important subsectors in $N = 4$ SYM are the $SU(2)$, $SU(1|1)$ and $SU(1,1)$ subsectors. In these subsectors we consider, respectively, the following type of composite fields,

$$
O_{SU(2)} = \text{Tr} (Z^{J-M} X^M) + \ldots, \quad O_{SU(1|1)} = \text{Tr} (Z^{J-M} \Psi^M) + \ldots, \quad O_{SU(1,1)} = \text{Tr} (D^S Z^J) + \ldots
$$

where the dots stand for permutations of the several constituent fields. For the $SU(2)$ and $SU(1|1)$ sectors the dilatation operators mixing these states (and whose eigenvalues are the anomalous dimensions of the corresponding eigenvectors) is given by

$$
H = \sum_{n=1}^{L} (1 - P_{n,n+1}) , \quad \text{for } SU(2) \text{ and } SU(1|1)
$$

In the $SU(2)$ sector $P$ is the permutation operator

$$
P_{12}|ZZ\ldots) = |ZZ\ldots)
$$

while in the $SU(1|1)$ sector it is the super-permutatation operator

$$
P_{12}|ZZ\ldots) = |ZZ\ldots)
$$

In the $SU(1,1)$ sector we represent the operators

$$
\text{Tr}((D^{S_1} Z)(D^{S_2} Z)(D^{S_3} Z)\ldots) \quad |S_1, S_2, S_3, \ldots) , \quad S_j = 0, 1, 2, \ldots
$$

Then the dilatation operator (or spin chain Hamiltonian) reads $H = \sum_{n=1}^{L} H_{n,n+1}$ where

$$
H_{n,n+1}|\ldots, S_n, S_{n+1}, \ldots) = \sum_{k=0}^{S_n+S_{n+1}} \left( \delta_{k,S_n} (h(S_n) + h(S_{n+1})) - \frac{1 - \delta_{k,S_n}}{|S_n - k|} \right) |\ldots, k, S_n+S_{n+1} - k, \ldots),
$$

and $h(j) = \sum_{k=1}^{j} \frac{1}{k}$ are the harmonic numbers. The goal of this exercise is to explore these Hamiltonians and in particular to compute the magnon S-matrix between fundamental excitations in the difference sectors.
Revisiting (and refining) the $SU(2)$ lecture

The current section is independent from the next two sections.

1) The action of the Hamiltonian (1.2) was implemented in the lecture. Refine this action so that we need not specify the length $L$ of the state we are acting on as an argument of the Hamiltonian.

**Hint:** Introduce a function which acts on a generic state and yields the length of the spin chain and then you can recycle the previous Hamiltonian.

2) We computed the anomalous dimension of the Konishi state by using one single representative for equivalent states (see function `CanOrder`). Alternatively one could symmetrize the original state over all cyclic permutations. Check that this yields indeed the same result for the anomalous dimension of the Konishi state.

3) When acting on single magnon excitations we obtained, at some point, the output

\[
(-1 + e^{ikL_0}) \left( e^{ik} |ZZ \ldots ZZX \rangle - |XZZ \ldots ZZ \rangle \right)
\]

(where in the lecture $L_0$ was equal to 9). This was zero because $\frac{kl_0}{2\pi} \in \mathbb{Z}$. How could you use `Assumptions` to automatically get a zero result?

The $SU(1|1)$ sector

4) Repeat the analysis done in the $SU(2)$ sector in the lecture for the $SU(1|1)$ Hamiltonian, namely

- Compute the dispersion relation $\epsilon(p)$ of the single magnon $\sum_{j=1}^{n} e^{ipn} |ZZ \ldots Z \Psi Z \ldots Z \rangle_{n-1, L-n}$

- Compute the scattering matrix between two magnons

- Check that the 3-magnon Bethe ansatz wave function is indeed an eigenvector of the spin chain Hamiltonian

**Hint:** The S-matrix you should find is the simplest you can imagine.

The $SU(1, 1)$ sector

4) Repeat the analysis done in the $SU(2)$ sector in the lecture for the $SU(1, 1)$ Hamiltonian, namely

- Compute the dispersion relation $\epsilon(p)$ of the single magnon $\sum_{j=1}^{n} e^{ipn} |ZZ \ldots D Z \ldots Z \rangle_{n-1, L-n}$

- Compute the scattering matrix between two magnons

- Check that the 3-magnon Bethe ansatz wave function is indeed an eigenvector of the spin chain Hamiltonian

**Hint:** The S-matrix you should find is (surprisingly!) almost the same as the one found when studying the $SU(2)$ Hamiltonian. What is the difference?

**Warning:** Notice that particles in this model are not hardcore but instead can superpose at the same site. This leads to some slightly non-trivial modifications in the code.